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**Decay laws for resonances
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unstable eigenvalues close to a threshold**

by

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Decay laws for resonances produced by perturbation of unstable eigenvalues close to a threshold

Dedicated to Ari Laptev on the occasion of his 60th birthday

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We report results concerning the decay laws for resonances produced by perturbation of unstable bound states close to a threshold. The model Hamiltonian is of the form

$$H_\varepsilon = \begin{bmatrix} H_{\text{op}} & 0 \\ 0 & E_0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & W_{12} \\ W_{21} & 0 \end{bmatrix} \quad \text{on } \mathcal{H} = \mathcal{H}_{\text{op}} \oplus \mathbf{C},$$

appearing in the study of Feshbach resonances. The operator H_{op} is assumed to have the properties of a Schrödinger operator in odd dimensions, with a threshold at zero. We consider for ε small the survival probability $|\langle \Psi_0, e^{-itH_\varepsilon} \Psi_0 \rangle|^2$, where Ψ_0 is the eigenfunction corresponding to E_0 for $\varepsilon = 0$. For E_0 in a small neighborhood of the origin *independent of* ε , the survival probability amplitude is expressed in terms of some special functions related to the error function, up to error terms vanishing as $\varepsilon \rightarrow 0$. This allows for a detailed study of the crossover from exponential to non-exponential decay laws, and then to the bound state regime, as the position of the resonance is tuned across the threshold.

Keywords: Decay law, non-exponential decay, Fermi Golden Rule

1. Introduction

This article is concerned with the decay laws for resonances produced by perturbation of unstable bound states. The problem has a long and distinguished history in quantum mechanics, and there is an extensive body of literature about decay laws for resonances in general, both at the level of theoretical physics (see

e.g. [10, 11, 23, 27–29] and references therein), and at the level of rigorous mathematical physics (see e.g. [3–6, 9, 13, 17–19, 25, 31, 32] and references therein). It started with the computation by Dirac of the decay rate in second order time-dependent perturbation theory, leading to the well known exponential decay law, $e^{-\Gamma t}$. Here Γ is given by the famous “Fermi Golden Rule” (FGR), $\Gamma \sim |\langle \Psi_0, \varepsilon W \Psi_{\text{cont}, E_0} \rangle|^2$, where Ψ_0 , E_0 are the unperturbed bound state eigenfunction and energy, respectively, and Ψ_{cont, E_0} is the continuum “eigenfunction”, degenerate in energy with the bound state. The FGR formula met with a fabulous success, and as a consequence, the common wisdom is that the decay law for the resonances produced by perturbation of non-degenerate bound states is exponential, at least in the leading non-trivial order in the perturbation strength.

However, it has been known for a long time, at least for semi-bounded Hamiltonians, that the decay law cannot be purely exponential; there must be deviations at least at short and long times. This implies that, in more precise terms, the question is whether the decay law is exponential up to errors vanishing as the perturbation strength tends to zero. So at the rigorous level the *crucial problem* is the *estimation of the errors*. This proved to be a hard problem, and only during the past decades consistent rigorous results have been obtained. The generic result is that (see [3, 4, 13, 17, 25] and references therein) the decay law is indeed (quasi)exponential, i.e. exponential up to error terms vanishing in the limit $\varepsilon \rightarrow 0$, as long as the resolvent of the unperturbed Hamiltonian is sufficiently smooth, when projected onto the subspace orthogonal to the eigenvalue under consideration. For most cases of physical interest this turns out to be the case, as long as the unperturbed eigenvalue lies in the continuum, far away from the threshold energies.

The problem with the exponential decay law appears for bound states situated near a threshold, since in this case the projected resolvent might not be smooth, or may even blow up, when there is a zero resonance at the threshold, see e.g. [15–17] and references therein. As it has been pointed out in [2, 12] at threshold the FGR formula does not apply. Moreover, the fact that the non-smoothness of the resolvent opens the possibility of a non-exponential decay at all times has been mentioned at the heuristic level [21, 23].

Let us mention that the question of the decay law for near threshold bound states is more than an academic one. While having the bound state in the very neighborhood of a threshold is a non-generic situation, recent advances in experimental technique have made it possible to realize this case for the so-called Feshbach resonances, where (with the aid of a magnetic field) it is possible to tune the energy of the bound state (and then the resonance position) throughout a neighborhood of the threshold energy.

The decay law for the case, when the resonance position is close to the threshold, has been considered at the rigorous level in [17–19], but only under the condition that the shift in the energy due to perturbation is sufficiently large, such that the resonance position is at a distance of order ε from the threshold. In this case it turns out that the decay law is still exponential, but the FGR has to be modified.

In this paper we report rigorous results for the case, when the resonance position is *anywhere* in a small ε -*independent* neighborhood of the threshold. To approximate the survival probability amplitude we use an appropriate ansatz, close in the spirit to the well known Lorentzian approximation for perturbed eigenvalues far from the threshold, but with a functional form taking into account the behavior of the resolvent near the threshold. The main technical result is the control of the error due to this approximation. The approximated survival probability amplitude is expressed in terms of some special functions, related to the error function, replacing the exponential function in the decay law. As a result, we are able to obtain a rigorous and detailed description of the crossover of the decay law, as the resonance position is tuned through the threshold from positive to negative energies via tuning of E_0 : Exponential decay with the usual FGR decay rate, to exponential decay with the modified FGR decay rate, then to non-exponential decay, and finally to bound state behaviour.

In what follows we present the guiding heuristics discussion, and some of the main results. The proofs, additional results, and other details are contained in [7, 8].

2. Generalities

We develop the theory in a somewhat abstract setting, which is applicable to two channel Schrödinger operators in odd dimensions, as they appear for example in the theory of Feshbach resonances (see e.g. [22, 30], and references therein).

Consider

$$H = \begin{bmatrix} H_{\text{op}} & 0 \\ 0 & H_{\text{cl}} \end{bmatrix} \quad \text{on } \mathcal{H} = \mathcal{H}_{\text{op}} \oplus \mathcal{H}_{\text{cl}}.$$

In concrete cases $\mathcal{H}_{\text{op}} = L^2(\mathbf{R}^3)$ (or $L^2(\mathbf{R}_+)$ in the spherically symmetric case), and $H_{\text{op}} = -\Delta + V_{\text{op}}$ with $\lim_{|\mathbf{x}| \rightarrow \infty} V_{\text{op}}(\mathbf{x}) = 0$. H_{op} describes the “open” channel. As for the “closed” channel, one starts again with a Schrödinger operator, but with $\lim_{|\mathbf{x}| \rightarrow \infty} V_{\text{cl}}(\mathbf{x}) = V_{\text{cl},\infty} > 0$. One assumes that H_{cl} has bound states below $V_{\text{cl},\infty}$, which may be embedded in the continuum spectrum of H_{op} . Only these bound states are relevant for the problem at hand. Thus one can retain only one isolated eigenvalue (or a group of almost degenerate eigenvalues isolated from the rest of the spectrum); the inclusion of the rest of the spectrum of H_{cl} merely “renormalizes” the values of some coefficients, without changing the qualitative picture. In this paper we shall consider only non-degenerate eigenvalues, i.e. we shall take $H_{\text{cl}} = E_0$ in $\mathcal{H}_{\text{cl}} = \mathbf{C}$, such that

$$H = \begin{bmatrix} H_{\text{op}} & 0 \\ 0 & E_0 \end{bmatrix}, \tag{1}$$

on

$$\mathcal{H} = \mathcal{H}_{\text{op}} \oplus \mathbf{C} = \left\{ \Psi = \begin{bmatrix} \psi \\ \beta \end{bmatrix} \mid \psi \in \mathcal{H}_{\text{op}}, \beta \in \mathbf{C} \right\}.$$

In addition to the spectrum of H_{op} the operator H has a bound state $\Psi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ at E_0 .

The problem is to study the fate of E_0 , when an interchannel perturbation

$$\varepsilon W = \varepsilon \begin{bmatrix} 0 & W_{12} \\ W_{21} & 0 \end{bmatrix}, \quad \varepsilon > 0, \quad (2)$$

is added to H , i.e. the total Hamiltonian is

$$H_\varepsilon = H + \varepsilon W. \quad (3)$$

We assume that W is a bounded self-adjoint operator.

The quantity to be studied is the so-called survival probability amplitude

$$A_\varepsilon(t) = \langle \Psi_0, e^{-itH_\varepsilon} \Psi_0 \rangle. \quad (4)$$

As in [17–19] we shall follow the approach in [12, 31] to write down a workable formula for $A_\varepsilon(t)$, i.e. we use the Stone formula to express the compressed evolution in terms of the compressed resolvent, and then we use the Schur-Livsic-Feshbach-Grushin (SLFG) partition formula to express the compressed resolvent as an inverse. One arrives at the following basic formula for $A_\varepsilon(t)$,

$$A_\varepsilon(t) = \lim_{\eta \searrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-itx} \text{Im } F(x + i\eta, \varepsilon)^{-1} dx \quad (5)$$

with

$$F(z, \varepsilon) = E_0 - z - \varepsilon^2 g(z), \quad g(z) = \langle \Psi_0, W Q^* (H_{\text{op}} - z)^{-1} Q W \Psi_0 \rangle. \quad (6)$$

Here Q is the projection onto the orthogonal complement of the bound state Ψ_0 , and it is viewed as an operator from \mathcal{H} to \mathcal{H}_{op} .

Since we are interested in the form of $A_\varepsilon(t)$, when E_0 is near a threshold of H_{op} , we shall assume that 0 is a threshold of H_{op} , and that E_0 is close to zero. To insure nontriviality we require $QW\Psi_0 \neq 0$.

Assumption 2.1.

- (i) There exists $a > 0$, such that $(-a, 0) \subset \rho(H_{\text{op}})$ (the resolvent set) and $[0, a] \subset \sigma_{\text{ess}}(H_{\text{op}})$.
- (ii) $|E_0| \leq \frac{1}{2}$.

From Assumption 2.1 and Eq. (6) one gets:

Proposition 2.1.

- (i) $g(z)$ is analytic in $\mathbf{C} \setminus \{(-\infty, -a] \cup [0, \infty)\}$.
- (ii) $g(z)$ is real and strictly increasing on $(-a, 0)$.
- (iii) $\text{Im } g(z) > 0$ for $\text{Im } z > 0$.

Since we are interested in the case, where E_0 is tuned past the threshold, we need assumptions about the behavior of the function $g(z)$ in a neighborhood of the origin.

Assumption 2.2. For $\operatorname{Re} \kappa \geq 0$ and $z \in \mathbf{C} \setminus [0, \infty)$ we let $\kappa = -i\sqrt{z}$. Let for $a > 0$, $D_a = \{z \in \mathbf{C} \setminus [0, \infty) \mid |z| < a\}$. Then for $z \in D_a$

$$g(z) = \sum_{j=-1}^4 \kappa^j g_j + \kappa^5 r(\kappa), \quad \frac{d}{dz} g(z) = -\frac{1}{2\kappa} \sum_{j=-1}^4 j \kappa^{j-1} g_j + \kappa^3 s(\kappa), \quad (7)$$

where $\sup_{z \in D_a} \{|r(\kappa)|, |s(\kappa)|\} < \infty$. Furthermore, we assume that $\lim_{\operatorname{Im} z \searrow 0} (g(z) - g_{-1} \kappa^{-1})$ exists and is continuous on $(-a, a)$.

Notice that due to Proposition 2.1 (ii), the coefficients g_j are real. Assumption 2.2 includes the case, when $H_{\text{op}} = -\Delta + V_{\text{op}}$ in odd dimensions. For the expansions of the resolvent of $-\Delta + V_{\text{op}}$ leading to Eq. (7) the reader is sent to [14–18, 26, 33]. Examples of expansions with the corresponding explicit expressions for coefficients g_j are given in the Appendix to [17], with references to the literature.

Since the form of the decay law depends strongly upon the behaviour of $g(z)$ near 0, we divide the considerations into three cases.

- (1) The *singular* case, in which $g_{-1} \neq 0$. In the Schrödinger case this corresponds to the situation, when H_{op} has a zero resonance at the threshold (see e.g. [15, 17]). Let us recall that the free particle in one dimension belongs to this class. From Proposition 2.1(iii) follows that $g_{-1} > 0$.
- (2) The *regular* case, in which $g_{-1} = 0$ and $g_1 \neq 0$. We note that $g_{-1} = 0$ is the generic case for Schrödinger operators in one and three dimensions. Again from Proposition 2.1(iii) one has $g_1 < 0$. Let us remark that the behavior $\operatorname{Im} g(x+i0) \sim x^{1/2}$ as $x \rightarrow 0$ is nothing but the famous Wigner threshold law [24, 30].
- (3) The *smooth* case, in which $g_{-1} = g_1 = 0$. This case occurs for free Schrödinger operators in odd dimensions larger than three, and in the spherical symmetric case for partial waves $\ell \geq 1$, see [17, 18]. Notice that in this case $\frac{d}{dz} g(z)$ is uniformly bounded in D_a .

Throughout the paper H_{op} and W are kept fixed, while E_0 and ε are parameters. In stating the results we use the following notation:

- (i) $A \lesssim B$ means that there exists a constant c such that $A \leq cB$. An analogous definition holds for $A \gtrsim B$.
- (ii) $A \simeq B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.
- (iii) $A \cong B$ means that A and B are equal to leading order in a parameter, e.g. $A = B + \delta(\varepsilon)$ with $\lim_{\varepsilon \searrow 0} \delta(\varepsilon) = 0$.

3. Heuristics

For E_0 outside a small (possibly ε -dependent) neighborhood of the origin, the situation is well understood, both at the heuristic level, and at the rigorous level. Indeed,

for negative E_0 , using the analytic perturbation theory, one can show that

$$|A_\varepsilon(t) - e^{-itE_\varepsilon}| \lesssim \varepsilon^2, \quad (8)$$

where E_ε is the perturbed eigenvalue, which coincides with E_0 in the limit $\varepsilon \rightarrow 0$. As a consequence, the survival probability remains close to one uniformly in time.

On heuristic grounds, if E_0 is positive, i.e. embedded in the essential spectrum of H_{op} , Ψ_0 turns into a metastable decaying state. The main problem is to compute the “decay law”, i.e. $|A_\varepsilon(t)|^2$, up to error terms vanishing in the limit $\varepsilon \rightarrow 0$. For eigenvalues embedded in the continuum spectrum the heuristics for the exponential decay law $|A_\varepsilon(t)|^2 \cong e^{-2\Gamma(\varepsilon)t}$ runs as follows.

Suppose $F(z, \varepsilon)$ is sufficiently smooth, as z approaches the real line from above, $F(x + i0, \varepsilon)$, for x in a neighborhood of E_0 . Let $F(x + i0, \varepsilon) = R(x, E_0, \varepsilon) + iI(x, \varepsilon)$. Then the main contribution to the integral in Eq. (5) comes from the neighborhood of the point $x_0(E_0, \varepsilon)$ where $R(x, E_0, \varepsilon) = 0$ and in this neighborhood (to simplify the notation we omit in what follows the dependence of $R(x, E_0, \varepsilon)$, $I(x, \varepsilon)$, and $x_0(E_0, \varepsilon)$ on E_0 and ε).

$$F(x + i0, \varepsilon) \cong x_0 - x + iI(x_0), \quad (9)$$

and then

$$\text{Im } F(x, \varepsilon)^{-1} \cong \frac{-I(x_0)}{(x - x_0)^2 + I(x_0)^2}, \quad (10)$$

i.e. it has a Lorentzian peak shape leading to

$$|A_\varepsilon(t)|^2 \cong e^{-2|I(x_0)|t}. \quad (11)$$

In the cases where the resolvent has an analytic continuation through the positive semi-axis [12], the resonance is defined as the zero, $z_r = x_r + iy_r$, of $F(z, \varepsilon)$ situated near E_0 in the lower half plane. In the case, where we have smoothness, but not analyticity, we take $x_0 + iI(x_0)$ as the “resonance”. Using the form of $F(z, \varepsilon)$ given in Eq. (6) one can show that

$$|x_0 + iI(x_0) - z_r| \lesssim \varepsilon^2 |y_r|, \quad (12)$$

for ε sufficiently small. We note that the estimate Eq. (12) agrees with the general uniqueness result in [20].

The problem with the energies near the threshold is that $F(x + i0, \varepsilon)$ might not be smooth and can even blow up (see Assumption 2.2), if the open channel has a zero resonance at the threshold. Then a Lorentzian approximation might break down. For the case at hand, elaborating on a heuristic argument in [21], one can quantify at the heuristic level how far from the origin $x_0 > 0$ must be in order to have a chance for an exponential decay law: The contribution of the tail at negative x of the Lorentzian must be negligible. Since

$$\int_{-\infty}^0 \frac{|I(x_0)|}{(x - x_0)^2 + I(x_0)^2} dx \simeq \frac{|I(x_0)|}{x_0}, \quad (13)$$

one gets the condition

$$|I(x_0)| \ll x_0. \quad (14)$$

Consider first the condition Eq. (14) in the *singular* case. For $x > 0$ small enough

$$I(x) \cong -g_{-1}\varepsilon^2 x^{-1/2},$$

and the condition Eq. (14) gives $g_{-1}\varepsilon^2 x_0^{-1/2} \ll x_0$, i.e. $x_0 \gg \varepsilon^{4/3}$. If we take (by adjusting E_0 !) $x_0 = b\varepsilon^p$, then one obtains, for $0 \leq p < 4/3$, the exponential decay law (see Eq. (11))

$$|A_\varepsilon(t)|^2 \cong e^{-2g_{-1}b^{-1/2}\varepsilon^{2-p/2}t}. \quad (15)$$

Notice that for $p = 0$ (i.e. the resonance stays away from the threshold as $\varepsilon \rightarrow 0$), Eq. (15) is nothing but the usual Fermi Golden Rule (FGR) formula. However, for $p > 0$, but not very large (i.e. the resonance position approaches zero as $\varepsilon \rightarrow 0$, but not too fast), one gets a “modified FGR formula”, for which the ε -dependence of the resonance width is $\varepsilon^{2-p/2}$ instead of the usual ε^2 -dependence.

For the *regular* case, a similar argument leads to the condition

$$x_0 \gg \varepsilon^4, \quad (16)$$

and a decay law

$$|A_\varepsilon(t)|^2 \cong e^{-2|g_1|b^{1/2}\varepsilon^{2+p/2}t}. \quad (17)$$

Finally, in the *smooth* case the condition Eq. (14) reads

$$\varepsilon^2 x_0^{1/2} \ll 1, \quad (18)$$

which holds true irrespective of how close to zero x_0 is. In other words, in the smooth case one observes an exponential decay law (with a resonance width vanishing as $x_0 \rightarrow 0$), as the resonance position is tuned past the threshold, via the tuning of the eigenvalue E_0 .

For the regular and smooth cases the above heuristics are substantiated by the following two cases of the results in [17] (see also [4, 31]).

Theorem 3.1. (i) Assume that $F(z, \varepsilon)$ is $\frac{1}{2}$ -Hölder continuous uniformly for $z \in D_a$ and ε sufficiently small. Then for $|I(x_0)| \geq \text{const.}\varepsilon^\gamma$, $2 \leq \gamma < 4$,

$$|A_\varepsilon(t) - e^{-it(x_0 + iI(x_0))}| \lesssim \text{const.}\varepsilon^\delta, \quad \delta = 2 - \frac{\gamma}{2}. \quad (19)$$

(ii) Assume that $F(z, \varepsilon)$ is Lipschitz continuous uniformly for $z \in D_a$ and ε sufficiently small. Then

$$|A_\varepsilon(t) - e^{-it(x_0 + iI(x_0))}| \lesssim \varepsilon^2 |\ln \varepsilon|. \quad (20)$$

Indeed, the smooth case is covered by Theorem 3.1(ii). In the regular case, as far as $x_0 \gg \varepsilon^4$ (see Eq. (16)), we get from Eq. (6) and Eq. (7) that $|I(x_0)| \gg \varepsilon^4$, and one can apply Theorem 3.1(i).

However, neither the above heuristics nor previous results give any hint about the form of the decay law in the singular and regular case, when x_0 is very close to the threshold. Our result is that in this case the decay law is definitely non-exponential. Due to lack of space only results in the *regular case* are presented. The results in the singular case (which can be found in [7, 8]) are similar, in spite of the fact that the proofs are a bit more complicated, due to the singularity of $g(z)$ at threshold.

4. The model function, regular case

We recall first (see previous section) that in the case of embedded eigenvalues (i.e. $p = 0$) the “model function” approximating $F(z, \varepsilon)$ is the linear function $L(z) = \alpha + i\beta - z$, where the constants α and β are fixed by the condition that $F(x_0 + i0, \varepsilon) = L(x_0 + i0)$ i.e. F and L coincide at $x_0(\varepsilon)$.

In the regular case we replace $F(z, \varepsilon)$ for all $p \in (0, \infty)$ by the “model function”

$$H_r(z) = \alpha - z + \varepsilon^2 \beta \kappa, \quad (21)$$

resembling the expansion of $F(z, \varepsilon)$ around the threshold. The free parameters are fixed by the condition that F and H_r coincide at the zeroes of F . There are two real parameters α and β , to be determined. In the case $E_1 = E_0 - g_0 \varepsilon^2 \geq 0$ (when the zero of $R(x)$ is positive) the condition used is $F(x_0, \varepsilon) = H_r(x_0)$. In the case $E_1 < 0$ (when the zero, x_b , of $R(x)$ is negative and gives the energy of the bound state) the conditions used are $F(x_b, \varepsilon) = H_r(x_b)$ together with $\frac{d}{dx} F(x_b, \varepsilon) = \frac{d}{dx} H_r(x_b)$. Thus in the case $E_1 < 0$ our conditions determining α and β give as a result that the residues at the pole x_b of $\frac{1}{F(x, \varepsilon)}$ and $\frac{1}{H_r(x)}$ are equal.

5. Main results; error analysis

We are now in a position to formulate the main technical result: For ε sufficiently small and E_0 in an ε -independent neighborhood of the threshold, the error in $A_\varepsilon(t)$ due to the replacement of $F(z, \varepsilon)$ with the model function $H_r(z)$, as given by Eq. (21), vanishes in the limit $\varepsilon \rightarrow 0$.

Theorem 5.1. *Assume $g_{-1} = 0$, $g_1 \neq 0$. There exists c , $\frac{a}{2} \geq c > 0$, such that for sufficiently small ε , $|E_0| \leq c$, and all $t \geq 0$, we have*

$$\begin{aligned} & \left| A_\varepsilon(t) - \lim_{\eta \searrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-itx} \operatorname{Im} H_r(x + i\eta)^{-1} dx \right| \\ & \lesssim \begin{cases} \varepsilon^2 (1 + x_0^{1/2} |\ln \varepsilon|), & \text{for } E_1 > 0, \\ \varepsilon^2, & \text{for } E_1 \leq 0. \end{cases} \end{aligned} \quad (22)$$

6. Crossover from exponential to non-exponential decay laws

The contribution of the negative semi-axis in $\lim_{\eta \searrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-itx} \text{Im } H_r(x + i\eta)^{-1} dx$ is just the residue at the zero, x_b , of $H_r(z)$ (when it exists). Accordingly

$$\lim_{\eta \searrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-itx} \text{Im } H_r(x + i\eta)^{-1} dx = -\frac{1}{\frac{d}{dx} H_r(x_b)} e^{-itx_b} + \hat{A}_{\varepsilon,r}(t) \quad (23)$$

with

$$\hat{A}_{\varepsilon,r}(t) \equiv \frac{1}{\pi} \int_0^{\infty} e^{-itx} \text{Im } H_r(x + i0)^{-1} dx. \quad (24)$$

Since H_r has a simple functional form with only two free parameters, the integral in the r.h.s of Eq. (24) can be evaluated numerically or expressed in closed form in terms of special functions:

$$\mathcal{I}_1(z) = \frac{2z}{i\pi} \int_0^{\infty} \frac{e^{-ix^2}}{x^2 - z^2} dx, \quad \mathcal{I}_p(z) = \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \mathcal{I}_1(z), \quad p > 1, \quad (25)$$

closely related to the error function.

Passing to the variable $k = \sqrt{z} = i\kappa$ we get

$$\hat{A}_{\varepsilon,r}(t) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{e^{-itk^2}}{P_r(\kappa)} k dk, \quad P_r(\kappa) = \kappa^2 + \varepsilon^2 \beta \kappa + \alpha. \quad (26)$$

When the zeroes of P_r are distinct (the other case can be recovered as a limit) a partial fraction decomposition yields the following result.

Proposition 6.1.

$$\hat{A}_{\varepsilon,r}(t) = -\sum_{j=1}^2 q_j \mathcal{I}_1(i\kappa_j \sqrt{t}), \quad (27)$$

where

$$\kappa_j = \frac{1}{2} (-\varepsilon^2 \beta - (-1)^j \sqrt{\varepsilon^4 \beta^2 - 4\alpha}), \quad j = 1, 2, \quad (28)$$

are the roots of $P_r(\kappa)$, and

$$q_j = \frac{1}{2} (1 + (-1)^j \frac{\varepsilon^2 \beta}{\sqrt{\varepsilon^4 \beta^2 - 4\alpha}}), \quad j = 1, 2, \quad (29)$$

are the corresponding residues of $\frac{\kappa}{P_r(\kappa)}$.

As a consequence of Proposition 6.1 we can now discuss the various regimes.

The exponential regime

According to the heuristics, if we set $\alpha = b\varepsilon^p$, $b > 0$, then for $p \in [0, 4)$ the decay law is still exponential. Notice that as $\varepsilon \rightarrow 0$ we have $\varepsilon^2/\sqrt{\alpha} \simeq \varepsilon^{2-\frac{p}{2}}$.

Proposition 6.2. (i) For $p \in [0, 4)$:

$$\hat{A}_{\varepsilon,r}(t) = 2q_2 e^{i\kappa_2^2 t} - \frac{\beta}{2} \frac{\varepsilon^2}{\sqrt{\alpha}} \overline{\mathcal{I}_3(i\sqrt{\alpha t})} + \mathcal{O}(\varepsilon^{4-p}), \quad (30)$$

and up to error terms as in Theorem 5.1 we have $A_\varepsilon(t) = \hat{A}_{\varepsilon,r}(t)$.

(ii) For $p \in (0, 4)$ we have $|A_\varepsilon(t)|^2 = e^{-2|g_1|b^{1/2}\varepsilon^{2+\frac{p}{2}}t}$, up to errors vanishing as $\varepsilon \rightarrow 0$.

Note that the exponential law becomes less and less accurate due to the error term, as p approaches 4 from below. Thus one needs to compute corrections. Proposition 6.2 gives only the first order correction in $\varepsilon^2/\sqrt{\alpha}$, but the method of proof provides also the higher order corrections. Due to Eq. (6) and Eq. (7) we have (see [25] for the notation)

$$|g_1|b^{1/2}\varepsilon^{2+\frac{p}{2}} = |I(x_0)| = \pi\varepsilon^2 \langle \Psi_0, WQ^*\delta(H_{\text{op}} - x_0)QW\Psi_0 \rangle$$

to leading order in ε , and the exponential decay is correctly described by the “modified FGR”, in which one uses the standard formula, but computed at the position x_0 of the resonance, instead of at E_0 .

The bound state regime

If $\alpha = -b\varepsilon^p$, $b > 0$, then for $p \in [0, 4)$ one expects (see the heuristics) a bound state regime, i.e. to leading order the contribution comes from the bound state. The result below provides the mathematical substantiation as well as the first order correction. Again, the proof gives the means to compute higher order corrections.

As in the previous case, as $\varepsilon \rightarrow 0$, we have $\varepsilon^2/\sqrt{|\alpha|} \simeq \varepsilon^{2-\frac{p}{2}}$. Note that $\kappa_1 > 0$, and there is a contribution from the pole of $\frac{1}{H_r(z)}$ at $x_b = -\kappa_1^2$.

Proposition 6.3.

$$\hat{A}_{\varepsilon,r}(t) = -i\frac{\beta}{2} \frac{\varepsilon^2}{\sqrt{|\alpha|}} \mathcal{I}_3(i\sqrt{|\alpha|t}) + \mathcal{O}(\varepsilon^{(4-p)}), \quad (31)$$

and up to error terms as in Theorem 5.1 we have

$$A_\varepsilon(t) = \hat{A}_{\varepsilon,r}(t) + \left(1 - i\frac{\beta}{2} \frac{\varepsilon^2}{\sqrt{|\alpha|}}\right) e^{i\kappa_1^2 t}. \quad (32)$$

To leading order one obtains the bound state behavior $|A_\varepsilon(t)|^2 = 1$.

The non-exponential regime

We come now to the most interesting part of our analysis, when

$$|\alpha| = b\varepsilon^p, \quad b > 0, \quad \text{with } p \geq 4. \quad (33)$$

According to the heuristics, for these values of p the decay law is neither (quasi)-exponential nor bound state like. We consider two cases separately.

Case 1: The threshold regime given by $p > 4$.

In this case the survival probability amplitude is given by

Proposition 6.4. *Up to errors as in Theorem 5.1 we have*

$$A_\varepsilon(t) = e^{is}(1 - \operatorname{erf}(e^{i\pi/4}s^{1/2})) + \mathcal{O}(\varepsilon^{p-4}), \quad (34)$$

where $\operatorname{erf}(u)$ is the usual error function [1] and $s = \beta^2\varepsilon^4t$.

The result Eq. (34) implies that the decay law is non-exponential for all $p > 4$, and that the leading term is independent of α and equals the threshold case $x_0 = 0$. It follows also that the threshold decay time scale in the regular case is $t \sim \varepsilon^{-4}$ for all $p > 4$. Note that the “modified FGR” gives a decay time scale $t \sim \varepsilon^{-(2+\frac{p}{2})}$, such that in this case it breaks down even at the time scale level.

Case 2: The “crossover regime”, which is given by $p = 4$.

In this case, in scaled variables $s = \varepsilon^4\beta^2t$, $f = \frac{\alpha}{\varepsilon^4\beta^2}$ (note that for $p = 4$, $f = \text{const.}$), we have directly from Eq. (24) and Eq. (21):

$$\hat{A}_\varepsilon(t) = \frac{1}{\pi} \int_0^\infty \frac{y^{1/2}}{(f-y)^2 + y} e^{-isy} dy, \quad (35)$$

and the integral has been analyzed numerically in [7]. The decay law is non-exponential for finite f , while as $f \rightarrow \pm\infty$, one reaches the exponential and bound state behaviour, respectively.

We illustrate the two cases discussed above. In case 1 a plot of the leading term in $|A_\varepsilon(t)|^2$ is given in Fig. 1. In Fig. 2 we have illustrated case 2, for negative values of the parameter f . In Fig. 3 the same for positive values of f . In Fig. 4 we have taken $f = 24$. It is clear from this logarithmic plot that the decay law is exponential initially.

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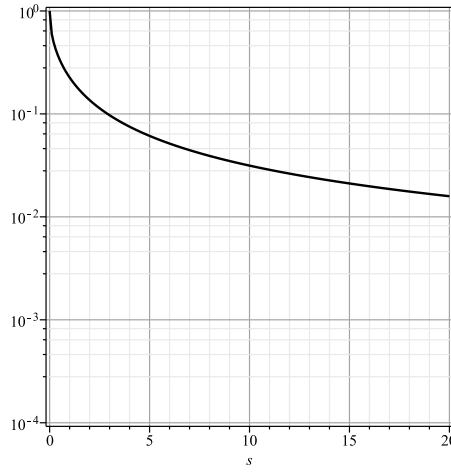


Fig. 1. Plot of the leading term in $|A_\varepsilon(t)|^2$ from Proposition 6.4. Vertical scale is logarithmic

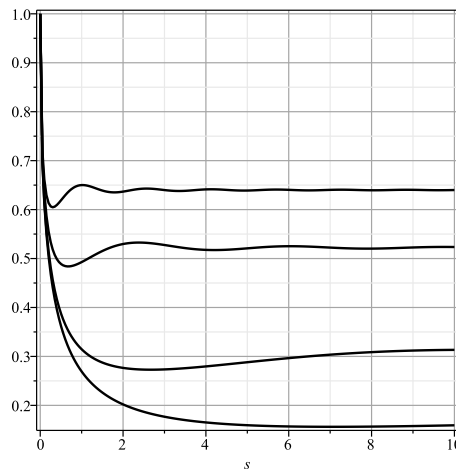


Fig. 2. Plot of the leading term in $|A_\varepsilon(t)|^2$ in case 2, $f = -6, -3, -1, -0.5$, from top to bottom. Vertical scale is linear

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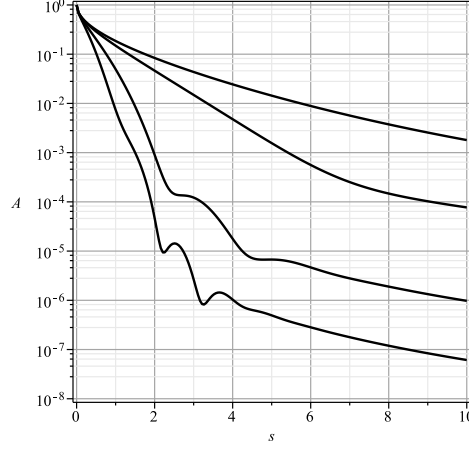


Fig. 3. Plot of the leading term in $|A_\varepsilon(t)|^2$ in case 2, $f = 0.5, 1, 3, 6$ from top to bottom. Vertical scale is logarithmic

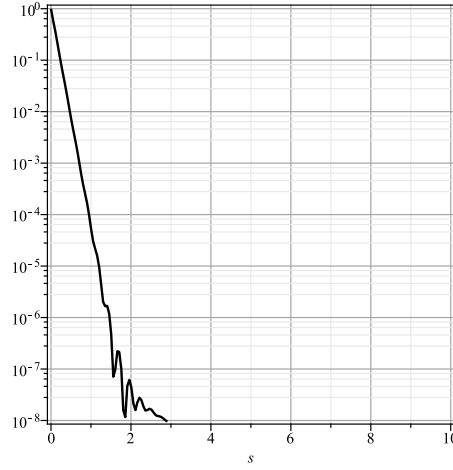


Fig. 4. Plot of the leading term in $|A_\varepsilon(t)|^2$ in case 2, $f = 24$. Vertical scale is logarithmic

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